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ON THE NUMBER OF GRAPHS WITHOUT 4-CYCLES

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The number F_n of n-vertex graphs lacking a 4-cycle is shown to satisfy $\frac{1}{2}n^{3/2} \leq \log F_n \leq 1.08192n^{3/2}$ for large n. This represents an improvement of the previously known upper and lower bounds and resolves a long-standing question posed by Erdös. The result is obtained by using an argument previously used by the authors to bound the numbers of lattices on n elements.

Introduction

In this paper we consider the number F_n of n-vertex labelled graphs without loops or multiple edges in which there is no four vertex cycle. We first give a construction which produces $(\sqrt{2})^{n^{3/2}}$ graphs of the desired type. It is not hard to find an upper bound of the form $n^{cn^{3/2}}$ (c a constant) for F_n.

The question as to which of these two expressions is closer to F_n has been open. Erdös some years ago posed the question: is the logarithm of F_n asymptotic to (or bounded in) both directions by a constant multiple of $n^{3/2}$, or by a constant multiple of $(\log n)n^{3/2}$? In this note we resolve this question, showing that the logarithm of F_n lies between two multiples of F_n, as follows:

$$\tfrac{1}{2}n^{3/2} \leq \log_2 F_n + \mathrm{o}(n^{3/2}) \leq (1.081919\ldots)n^{3/2}.$$

The problem is closely related to that of enumerating lattices on n elements. One important class of lattices are those in which chains have no more than four elements; that is, in which there are (in addition to the minimum and maximum elements) only atoms and co-atoms. Kleitman and Winston [2] recently found an upper bound of the form $2^{cn^{3/2}}$ (c a constant) for the number of such lattices.

An undirected bipartite graph in which there are no 4-cycles can be taken as the graph of the covering relation between atoms and co-atoms of such a lattice. The bound on the number of these lattices could then be applied iteratively to

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yield a bound of the desired form for general 4-cycle-free graphs. However, a better upper bound can be obtained by applying the argument directly to graphs.

The lower bound

We use a property of projective planes admitting polarities to construct graphs of the desired type. A projective plane of order q has a (point, line) incidence matrix $A=(a_{ij})$ in which a_{ij} is zero if point i does not lie on line j, and a_{ij} is one if point i does lie on line j. A is a (q^2+q+1) by (q^2+q+1) matrix, and every row and column sum is $q+1$. Furthermore, for every pair of distinct rows i and j of A, there is exactly one k with $a_{ik}=a_{jk}=1$; the dual property holds for columns.

A polarity of a projective plane is an incidence-preserving one-to-one map of order 2 between points and lines and between lines and points. If a plane admits a polarity, we can assume that point i is mapped into line i. This means that the incidence matrix A of such a plane is symmetric.

We construct from such a matrix A a labelled graph G with $n=q^2+q+1$ vertices by including an edge between vertex i and vertex j if and only if $i\neq j$ and $a_{ij}=a_{ji}=1$. Clearly G has no 4-cycles.

Furthermore, any subgraph of G lacks 4-cycles. There are at least $q(q^2+q+1)$ ones off the main diagonal of A, so there are at least $\frac{1}{2}n(\sqrt{n-\frac{3}{4}}-\frac{1}{2})$ edges in G. Thus

$$\log_2 F_n \geqslant \tfrac{1}{2}n^{3/2}+c(n^{3/2}) \tag{1}$$

if there is a projective plane of order q admitting a polarity, with $n=q^2+q+1$.

Finally note that there is a Desarguian projective plane of every prime power order. Dembowski [1, p. 157] has shown that every Desarguian plane admits a polarity. For large n, the difference between n and the nearest prime power is small compared to n, so (1) holds for all n.

The upper bound

Let d_i be the degree of the ith vertex of an n-vertex graph G lacking 4-cycles. Then $\sum_{i=1}^{n}\binom{d_i}{2}$ counts the number of pairs of vertices both connected to a third vertex. Thus

$$\sum_{i=1}^{n}\binom{d_i}{2}\leqslant\binom{n}{2}; \tag{2}$$

otherwise some pair of vertices would both be connected to two other vertices, giving a 4-cycle. This leads to

$$\frac{1}{n}\sum_{i=1}^{n}(d_i^2-d_i)\leqslant n-1, \tag{3}$$

and therefore to

$$\left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right) \leq n-1. \tag{4}$$

Solving the quadratic equation gives

$$\frac{1}{n}\sum_{i=1}^{n} d_i \leq \sqrt{n-\tfrac{3}{4}}+\tfrac{1}{2}. \tag{5}$$

The minimum degree $d\,(=\min d_i)$ in an n-vertex graph without 4-cycles is therefore at most $[\sqrt{n}]$; the greatest integer in the right hand side of (5) for large n.

Now note that F_n is at most n times the number of n-vertex graphs lacking 4-cycles in which the nth vertex is of minimum degree d. There are at most

$$\binom{n-1}{d} \leq \binom{n-1}{[\sqrt{n}]}$$

ways to add an nth vertex of degree d to an $(n-1)$-vertex graph lacking 4-cycles; thus

$$F_n \leq nF_{n-1}\binom{n-1}{[\sqrt{n}]} \leq F_{n-1}n^{\sqrt{n}}, \tag{6}$$

which gives the bound

$$\log_2 F_n \leq \tfrac{2}{3}(\log_2 n - \tfrac{2}{3})n^{3/2} + o(n^{3/2}) \tag{7}$$

when applied inductively.

We now demonstrate how to improve this upper bound so that the $\log_2 n$ term on the right hand side of (7) is replaced by a constant. We will prove:

Theorem. *The number F_n of graphs with n vertices in which there is no 4-cycle satisfies*

$$\log_2 F_n \leq \alpha n^{3/2} + o(n^{3/2}),$$

where

$$\alpha = \tfrac{2}{3}\max_{0\leq x\leq 1}\left(\frac{x^2\log_2 x^2 - (1-x^2)\log_2(1-x^2)}{x}\right)n \sim 1.08192$$

We preface the proof with a discussion of the problem and a description of the argument.

We will obtain the upper bound by providing a construction procedure to add a vertex to an $(n-1)$-vertex graph lacking 4-cycles so that the resulting graph also lacks 4-cycles. Each n-vertex graph lacking 4-cycles can be produced by applying this procedure to some $(n-1)$-vertex graph lacking 4-cycles.

We will show that the factor of $\binom{n-1}{[\sqrt{n}]}$ appearing in (6) can be replaced by a factor that is closer to $\binom{n/d}{d}$ for some number d; this factor attains its maximum when $d \sim \sqrt{n}/2.03$ and leads to the bound in the theorem.

The factor $\binom{n-1}{\sqrt{n}}$ was introduced in the inequality (6) above as a rough bound on the number of ways in which we could choose connections for a new minimum degree vertex. However, every time we choose a vertex with which the nth vertex (the vertex being added) is to share an edge, certain other vertices become ineligible to share an edge with the nth vertex because such an edge would complete a 4-cycle. This cuts down on the number of choices that can be made. In fact, it will turn out that after connecting an appropriately small number of vertices to the nth vertex, we can insure that there are at most n/d (d the minimum degree of the graph) vertices remaining which can share an edge with the nth vertex without completing a 4-cycle. This will provide the desired bound.

Proof of the Theorem. Let G_0 be the $(n-1)$-vertex graph G with an isolated nth vertex added. We proceed in d steps, adding an edge at each step. At the first step all vertices except the nth are eligible to be connected (i.e. to share an edge with) the nth vertex.

At each step, form an ordered list of the eligible vertices as follows: After $(i-1)$ vertices have been put on the list, determine which one of the remaining eligible vertices is connected in G by a path of length 2 to the greatest number of eligible vertices not yet on the list. (Ties can be broken arbitrarily.) That vertex will be ith on the list.

We then choose an eligible vertex from the list and add the edge between the nth vertex and the chosen vertex. All vertices which came before the chosen vertex are made ineligible, as are all vertices connected to the chosen vertex by a path of length 2. We then go to the next step, repeating until we either reach a dead end or form a graph in which the nth vertex has degree d.

Note that any n-vertex graph lacking 4-cycles (in which the nth vertex has minimum degree) can be formed from some $(n-1)$-vertex graph lacking 4-cycles with this procedure. The procedure provides a framework for producing these graphs; it does not limit the graphs that can be produced.

Suppose that there are s eligible vertices at some step and one vertex is connected at that step to the ith vertex on the list. We want to bound the number of eligible vertices left on the list after this choice. Let $\bar{s}=s-i+1$; there are $\bar{s}$ vertices after (and including) the chosen vertex on the list. Let q_m be the number of these $\bar{s}$ vertices with which the mth vertex of G shares an edge. Note that

$$\sum_{m=1}^{n-1} q_m \geqslant d\bar{s} \tag{8}$$

since all vertices have degree at least d. There are $\sum_{m=1}^{n-1} \binom{q_m}{2}$ pairs of the $\bar{s}$ vertices connected by paths of length 2. By again making use of the fact that the average of a square is greater than the square of its average we may bound this number of pairs according to

$$\sum_{m=1}^{n-1} \binom{q_m}{2} \geq \frac{d\bar{s}(d\bar{s}-n+1)}{2(n-1)}. \tag{9}$$

The chosen vertex is connected by a path of length 2 to the most others below it on the list. Therefore it is connected to at least

$$(2/\bar{s}) \sum_{m=1}^{n-1} \binom{q_m}{2} \geq \frac{d(d\bar{s}-n+1)}{(n-1)} \tag{10}$$

vertices further down the list by a path of length 2.

We can now bound the number of eligibles remaining on the list. There are at most

$$\bar{s} - \frac{d(d\bar{s}-n+1)}{(n-1)} \leq \bar{s}\left(1-\frac{d^2}{n}\right)+d \tag{11}$$

eligibles remaining so that $\bar{s}-n/d$ after one iteration becomes reduced to $(\bar{s}-n/d)(1-d^2/n)$. Applying (11) repeatedly from the first step where $\bar{s} \leq n-1$, we see that after z steps there are at most $n(1-d^2/n)^z+n/d$ and hence at most $\exp(-zd^2/n)+n/d$ remaining eligible vertices. Therefore if $zn/d^2 \geq \log n$ the number of eligibles from which choices can be made must be at most n/d.

(6) can therefore be altered to read

$$F_n \leq nF_{n-1}\binom{n}{z}\binom{n/d}{d-z}, \tag{12}$$

$$z=[(n/d^2\log n] \tag{13}$$

where $\binom{n}{z}$ overestimates the number of choices available in the early steps and $\binom{n/d}{d-z}$ is obtained by assuming free choice after there are only n/d eligible vertices left.

To prove the theorem it suffices to show that

$$\log_2\binom{n}{z}\binom{n/d}{d-z} < \tfrac{3}{2}\alpha\sqrt{n}, \tag{14}$$

(where α is given in the statement of the theorem). Note that if

$$d < \frac{\frac{3}{2}\alpha\sqrt{n}}{\log_2 n} \quad \text{then} \quad \log\binom{n}{d} < \tfrac{3}{2}\alpha\sqrt{n},$$

so we need only consider values of d obeying

$$d \geq \frac{\frac{3}{2}\alpha\sqrt{n}}{\log_2 n}.$$

This means we can take z to be at most the order of $(\log n)^3$, so the $\binom{n}{z}$ factor is negligible.

Using Sterling's formula, we have

$$\log\binom{n/d}{d-z} < \log_2\binom{n/d}{d} < -d\log_2\left(\frac{d^2}{n}\right) - \left(\frac{n}{d} - d\right)\log_2\left(1 - \frac{d^2}{n}\right). \tag{15}$$

Let $x^2 = d^2/n$. Then

$$\log_2\binom{n'}{d} < \sqrt{n}\,\frac{H(x^2)}{x}, \tag{16}$$

where $H(y) = -y\log_2 y - (1-y^2)\log_2(1-y^2)$ is the entropy function.

The right hand side of (16) attains a maximum of about 1.62288 when $x \approx 0.49143$. Thus

$$\log_2\binom{n/d}{d} < \tfrac{3}{2}(1.081919\ldots)\sqrt{n}, \tag{17}$$

proving the theorem.

The number of bipartite graph of order n is of the form c^{n^2}, so the number of graphs without odd cycles of any given size is of order c^{n^2}. The number of graphs lacking 6-cycles, or both 4 and 6-cycles, has not been investigated. The problem may well be amenable to the type of argument used here.

References

[1] P. Dembowski, Finite Geometries (Springer-Verlag, New York, 1968).

[2] D.J. Kleitman and K.J. Winston, The asymptotic number of lattices, Ann. Discrete Math. 6 (1980) 243–249.